# Propagation in Rectangular Waveguides with Arbitrary Internal and External Media

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***Abstract*—The propagation characteristics of electromagnetic waves in a rectangular waveguide of a homogeneous medium embedded in a different medium have been found approximately. The total field is assumed to consist of four crossing plane waves interconnected at the boundaries by reflection matrices. The method is more accurate than other approximate techniques. New results are presented for tunnel propagation and attenuation of degenerate modes in metallic waveguides.**

## I. INTRODUCTION

THE rectangular waveguide is one of the oldest transmission lines, so it is rather surprising that the subject has not been thoroughly dealt with under all possible circumstances. There are good reasons for this, though, since all components of the electric and magnetic fields are coupled to each other except for the nonrealizable case of perfect conductivity in the walls. For nearly all parts of the electromagnetic spectrum there is now a considerable interest in having more exact solutions for wave propagation in the general rectangular waveguide, i.e., with arbitrary internal and external media.

In the microwave region the dielectric surface waveguide or image line has been known for many years, but there has been a renewed interest recently in the optical version of this line [1], [15], the infrared version [3], and the millimeter or submillimeter version [4], [5]. At lower frequencies there is a need for a better understanding of the mode structure in tunnel propagation, although considerable progress in this area has recently been made [7], [8]. Except for two numerical solutions [2], [6] all the theoretical solutions have been restricted to the scalar (and solvable) two-dimensional case or slight modifications of it.

The exact field distribution in the two-dimensional case (parallel plate waveguide or dielectric slab) may be decomposed into two plane waves

$$A_1 \exp(-jk_x x) \exp(-jk_z z)$$

and

$$A_2 \exp(+jk_x x) \exp(-jk_z z)$$

where $z$ is along the waveguide axis, and $x$ is in the transverse direction. The two waves are connected at the boundaries through the reflection coefficients [16]. The rectangular waveguide may be viewed approximately as two slab waveguides, in which case at least four crossing plane waves are needed. We know that this simple field distribution is only an approximation, since additional corner fields, in general, will be needed to satisfy the boundary conditions. It is the purpose of this paper to present an approximate solution to the three-dimensional vector problem by invoking the reflection coefficients for the walls and assuming that only four crossing plane waves are present. The depolarization when a wave is reflected from a wall is taken into account. The system of equations turns out to be overdetermined and is solved in a least squares sense. The accuracy is determined by comparing the results with Schlosser and Unger [6] and some new results are presented for metallic waveguides and tunnels in lossy dielectrics.

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## II. REFLECTION FROM INFINITE PLANE INTERFACE

The reflection from a plane interface between two different media may easily be determined on the basis of the two Fresnel reflection coefficients $R^-$ and $R^+$. The derivation and notation are similar to those of Beckmann [9]. $R^-$ is usually referred to as the reflection coefficient for horizontal polarization, and $R^+$ is the corresponding coefficient for vertical polarization; the signs refer to the fact that $R^-$ tends towards minus one and $R^+$ towards plus one when the conductivity of the reflecting medium tends towards infinity.

In general, a wave changes polarization when reflected. It is by taking this depolarization into account that the present theory claims improved accuracy.

Referring to Fig. 1, we assume a wave with wave vector $\bar{k}_i$ incident on a plane interface with normal $\hat{n}$. The wave vector has three components $(k_x,k_y,k_z)k$ which may be arbitrary complex numbers satisfying the wave equation

$$k_x^2 + k_y^2 + k_z^2 = 1. \tag{1}$$

The time factor is $\exp(j\omega t)$ so the wave propagates as $\exp(-j\bar{k}_i \cdot \bar{r})$.

The electric field of the wave is bound to lie in a plane normal to $\bar{k}_i$, so in this plane we choose two orthogonal basis vectors, $\hat{e}^-$ and $\hat{e}^+$. We are free to choose one of them, and for the sake of convenience we assume that $\hat{e}^-$ lies

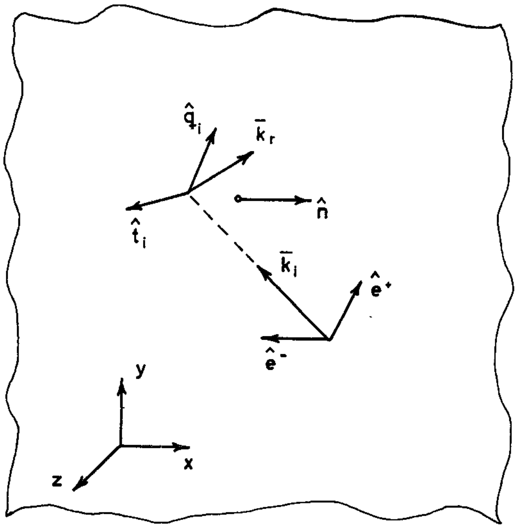


Fig. 1. Incident ($\bar{k}_i$) and reflected ($\bar{k}_r$) waves from a plane surface given by the normal $\hat{n}$.

in the $x$–$z$ plane, so

$$\hat{e}_i^- = \frac{\hat{y} \times \hat{k}_i}{|\hat{y} \times \hat{k}_i|} \tag{2}$$

and

$$\hat{e}_i^+ = \hat{k}_i \times \hat{e}_i^-. \tag{3}$$

Note that the basis system varies with the direction of $\hat{k}_i$. Two more unit vectors are introduced, $\hat{t}_i$ and $\hat{q}_i$, where $\hat{t}_i$ is a vector in the intersection between the surface and the phase front

$$\hat{t}_i = \frac{\hat{n} \times \hat{k}_i}{|\hat{n} \times \hat{k}_i|} \tag{4}$$

and $\hat{q}_i$ is normal to $\hat{t}_i$ and $\hat{k}_i$

$$\hat{q}_i = \hat{k}_i \times \hat{t}_i. \tag{5}$$

The component of $\bar{E}$ in the $\hat{t}_i$ direction is reflected with $R^-$ and the component in the $\hat{q}_i$ direction with $R^+$, where

$$R^- = \frac{k_1 \cos\theta - (k_2^2 - k_1^2 \sin^2\theta)^{1/2}}{k_1 \cos\theta + (k_2^2 - k_1^2 \sin^2\theta)^{1/2}} \tag{6}$$

$$R^+ = \frac{k_2^2 \cos\theta - k_1 (k_2^2 - k_1^2 \sin^2\theta)^{1/2}}{k_2^2 \cos\theta + k_1 (k_2^2 - k_1^2 \sin^2\theta)^{1/2}}. \tag{7}$$

$k_1$, $k_2$ are the wavenumbers in the media corresponding to the incident and the transmitted waves, respectively. The angle $\theta$ is the angle of incidence, defined by the relationship

$$\cos\theta = -\hat{n}\cdot\hat{k}_i. \tag{8}$$

It is in general a complex number.

The choice of branch cut in the square roots in $R^-$ and $R^+$ must be made on the basis of physical arguments. Usually the sign of the square root is chosen in this way:

$$\operatorname{Im}\,(k_2^2 - k_1^2 \sin^2\theta)^{1/2} \leq 0 \tag{9}$$

corresponding to a decay away from the interface of the transmitted wave. However, there is no reason why leaky waves should not be present; in fact, they are known to exist on dielectric slabs and cylinders, and we shall see later that they normally occur in tunnel propagation.

The direction $\hat{k}_r$ of the reflected wave is determined simply by $\hat{k}_i$ and $\hat{n}$, and unit vectors $\hat{e}_r^-, \hat{e}_r^+, \hat{t}_r, \hat{q}_r$ are defined in the same way as in (2)–(5).

Let us now assume an incident wave $\bar{E}_i = E_i^- \hat{e}_i^- + E_i^+ \hat{e}_i^+$. The reflected wave $\bar{E}_r = E_r^- \hat{e}_r^- + E_r^+ \hat{e}_r^+$ is connected with $\bar{E}_i$ through a reflection matrix $\bar{\bar{\Gamma}}$

$$\bar{E}_r = \bar{\bar{\Gamma}} \cdot \bar{E}_i \tag{10}$$

where the following expression for $\bar{\bar{\Gamma}}$ may be found by applying the preceding analysis:

$$\bar{\bar{\Gamma}} = \left\{ \begin{array}{c|c} R^- \hat{t}\cdot\hat{e}_r^- \hat{t}\cdot\hat{e}_i^- + R^+ \hat{q}_r\cdot\hat{e}_r^- \hat{q}_i\cdot\hat{e}_i^- & R^- \hat{t}\cdot\hat{e}_r^- \hat{t}\cdot\hat{e}_i^+ + R^+ \hat{q}_r\cdot\hat{e}_r^- \hat{q}_i\cdot\hat{e}_i^+ \\ \hline R^- \hat{t}\cdot\hat{e}_r^+ \hat{t}\cdot\hat{e}_i^- + R^+ \hat{q}_r\cdot\hat{e}_r^+ \hat{q}_i\cdot\hat{e}_i^- & R^- \hat{t}\cdot\hat{e}_r^+ \hat{t}\cdot\hat{e}_i^+ + R^+ \hat{q}_r\cdot\hat{e}_r^+ \hat{q}_i \hat{e}_i^+ \end{array} \right\} \tag{11}$$

where $\hat{t} = \hat{t}_i = \hat{t}_r$.

## III. MODAL EQUATIONS FOR RECTANGULAR WAVEGUIDES

The basic assumption of this paper is that the mode pattern (the field configuration which propagates without change of shape) in a rectangular waveguide may be closely approximated by the sum of four plane waves, these four waves being interconnected at the boundaries by the reflection matrices of Section II. The assumption will be justified by comparison with known results where available.

Thus we assume that the total electric field inside the waveguide is given by

$$\bar{E}_{tot} = \bar{A}_1 \exp(-j\bar{k}_1 \cdot \bar{r}) + \bar{A}_2 \exp(-j\bar{k}_2 \cdot \bar{r}) + \bar{A}_3 \exp(-j\bar{k}_3 \cdot \bar{r}) + \bar{A}_4 \exp(-j\bar{k}_4 \cdot \bar{r}) \quad (12)$$

where

$$\bar{k}_1 = (k_x, k_y, k_z) k_1$$
$$\bar{k}_2 = (k_x, -k_y, k_z) k_1$$
$$\bar{k}_3 = (-k_x, -k_y, k_z) k_1$$
$$\bar{k}_4 = (-k_x, k_y, k_z) k_1 \quad (13)$$

as indicated in Fig. 2. The $\bar{A}_i$ vector is a two-dimensional vector with unknown elements, in a basis-vector system belonging to $\bar{k}_i$ as indicated in the previous section. The projections $(k_x, k_y)$ are unknown, but $k_z$ is given by

$$k_z^2 = 1 - k_x^2 - k_y^2. \quad (14)$$

The boundary conditions are fulfilled in the following way on side 1. $\bar{A}_2$ and $\bar{A}_3$ are reflected waves corresponding to the incident waves $\bar{A}_1$ and $\bar{A}_4$, respectively. Using (11) we get for $y = b$

$$\bar{A}_2 \exp(-jk_xk_1x + jk_yk_1b - jk_zk_1z) = \bar{\bar{\Gamma}}_{11} \cdot \bar{A}_1 \exp(-jk_xk_1x - jk_yk_1b - jk_zk_1z) \quad (15)$$

or

$$\bar{A}_2 = \bar{\bar{\Gamma}}_{11} \exp(-j2k_yk_1b) \cdot \bar{A}_1 \quad (16)$$

where $\bar{\bar{\Gamma}}_{ij}$ is the reflection matrix from side $i$ with $\bar{k}_j$ incident. Similarly,

$$\bar{A}_3 = \bar{\bar{\Gamma}}_{14} \exp(-j2k_yk_1b) \cdot \bar{A}_4 \quad (17)$$
$$\bar{A}_1 = \bar{\bar{\Gamma}}_{24} \cdot \bar{A}_4 \quad (18)$$
$$\bar{A}_2 = \bar{\bar{\Gamma}}_{23} \cdot \bar{A}_3 \quad (19)$$
$$\bar{A}_1 = \bar{\bar{\Gamma}}_{32} \cdot \bar{A}_2 \quad (20)$$
$$\bar{A}_4 = \bar{\bar{\Gamma}}_{33} \cdot \bar{A}_3 \quad (21)$$
$$\bar{A}_3 = \bar{\bar{\Gamma}}_{42} \exp(-j2k_xk_1a) \cdot \bar{A}_2 \quad (22)$$
$$\bar{A}_4 = \bar{\bar{\Gamma}}_{41} \exp(-j2k_xk_1a) \cdot \bar{A}_1. \quad (23)$$

Using (19),(20),(21) all $\bar{A}$'s may be expressed in terms of $\bar{A}_3$, ending with the following matrix equation:

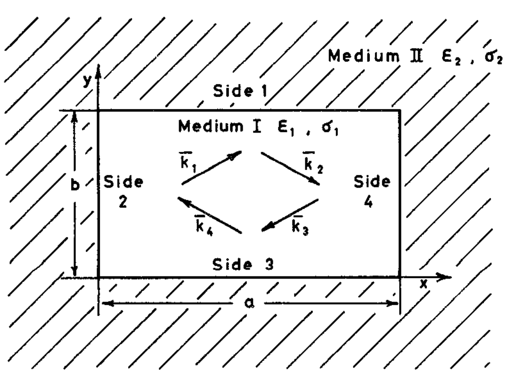


Fig. 2. Geometry of waveguide and $x$–$y$ components of the four basic wave vectors.

$$\bar{\bar{B}} \cdot \bar{A}_3 = \left\{ \begin{matrix} (\bar{\bar{U}} - \beta \bar{\bar{\Gamma}}_{11} \cdot \bar{\bar{\Gamma}}_{32}) \cdot \bar{\bar{\Gamma}}_{23} \\ \bar{\bar{U}} - \beta \bar{\bar{\Gamma}}_{14} \cdot \bar{\bar{\Gamma}}_{33} \\ \bar{\bar{\Gamma}}_{32} \cdot \bar{\bar{\Gamma}}_{23} - \bar{\bar{\Gamma}}_{24} \cdot \bar{\bar{\Gamma}}_{33} \\ \bar{\bar{U}} - \alpha \bar{\bar{\Gamma}}_{42} \cdot \bar{\bar{\Gamma}}_{23} \\ (\bar{\bar{U}} - \alpha \bar{\bar{\Gamma}}_{41} \cdot \bar{\bar{\Gamma}}_{24}) \cdot \bar{\bar{\Gamma}}_{33} \end{matrix} \right\} \cdot \bar{A}_3 = \bar{0} \quad (24)$$

where $\alpha = \exp(-j2k_xk_1a)$, $\beta = \exp(-j2k_yk_1b)$, and $\bar{\bar{U}}$ is the unit matrix. $\bar{\bar{B}}$ is a matrix with ten rows and two columns. This is the general case with four different media on the four sides. In the special case of only one external medium, $\bar{\bar{B}}$ can be simplified considerably as follows.

### *A. Reflection from Horizontal Walls*

Owing to the special choice of unit vectors in (2) and (3) $\Gamma_{1j}$ and $\Gamma_{3j}$ are especially simple, as

$$\cos\theta_h = k_y \quad (25)$$

for both side 1 and side 3. By inserting the proper incidence vectors in (11) it is found that

$$\bar{\bar{\Gamma}}_{11} = \bar{\bar{\Gamma}}_{14} = \bar{\bar{\Gamma}}_{32} = \bar{\bar{\Gamma}}_{33} = \bar{\bar{\Gamma}}_h = \left\{ \begin{matrix} R_h^- & 0 \\ 0 & R_h^+ \end{matrix} \right\} \quad (26)$$

(the subscript $h$ refers to horizontal walls).

### *B. Reflection from Vertical Walls*

In this case the angle of incidence is given by

$$\cos\theta_v = k_x \quad (27)$$

for both walls. By inserting the proper incidence vectors in (11) it is found that

$$\bar{\bar{\Gamma}}_{23} = \bar{\bar{\Gamma}}_{41} = \bar{\bar{\Gamma}}_v \quad (28)$$
$$\bar{\bar{\Gamma}}_{24} = \bar{\bar{\Gamma}}_{42} = \bar{\bar{\Gamma}}_v^T \quad (29)$$

where $T$ stands for transposed, $v$ for vertical, and

$$\bar{\bar{\Gamma}}_v = \left\{ \begin{matrix} R_v^+k_z^2 - R_v^-k_x^2k_y^2 & -(R_v^- + R_v^+)k_xk_yk_z \\ (R_v^- + R_v^+)k_xk_yk_z & R_v^-k_z^2 - R_v^+k_x^2k_y^2 \end{matrix} \right\} \cdot [(k_z^2 + k_y^2)(k_z^2 + k_x^2)]^{-1}. \quad (30)$$

Using these results $\bar{\bar{B}}$ may be simplified as

$$\bar{\bar{B}} = \left\{ \begin{matrix} (\bar{\bar{U}} - \beta \bar{\bar{\Gamma}}_h^2) \bar{\bar{\Gamma}}_v \\ \bar{\bar{U}} - \beta \bar{\bar{\Gamma}}_h^2 \\ \bar{\bar{\Gamma}}_h \bar{\bar{\Gamma}}_v - \bar{\bar{\Gamma}}_v^T \bar{\bar{\Gamma}}_h \\ \bar{\bar{U}} - \alpha \bar{\bar{\Gamma}}_v^T \bar{\bar{\Gamma}}_v \\ (\bar{\bar{U}} - \alpha \bar{\bar{\Gamma}}_v \bar{\bar{\Gamma}}_v^T) \bar{\bar{\Gamma}}_h \end{matrix} \right\} \quad (31)$$

with

$$\bar{\bar{B}} \cdot \bar{A}_3 = \bar{0}. \quad (32)$$

It is only in very special cases that (32) has exact solutions; in general, the right-hand side will be different from zero. A case which has a nontrivial solution is that of the perfectly conducting walls, where $R^+ = +1$, $R^- = -1$,

$$\bar{\bar{\Gamma}}_h^2 = \bar{\bar{U}} \tag{33}$$

$$\bar{\bar{\Gamma}}_h \bar{\bar{\Gamma}}_v = \bar{\bar{\Gamma}}_v^T \bar{\bar{\Gamma}}_h = -\bar{\bar{U}} \tag{34}$$

$$\bar{\bar{\Gamma}}_v^T \bar{\bar{\Gamma}}_v = \bar{\bar{\Gamma}}_v \bar{\bar{\Gamma}}_v^T = \bar{\bar{U}} \tag{35}$$

so $\bar{\bar{B}} = 0$ for $\alpha = \beta = 1$, independently of $\bar{A}_3$. This leads, of course, to the well-known relations

$$k_x k_1 a = n\pi \qquad k_y k_1 b = m\pi \tag{36}$$

for the standard, uncoupled waveguide modes. Another limiting case with exact solutions is the two-dimensional one, which may be achieved by having perfectly conducting (electric or magnetic) vertical walls,

$$1 - \beta (R_h^-)^2 = 0 \tag{37}$$

$$1 - \beta (R_h^+)^2 = 0 \tag{38}$$

which correspond to the standard dispersion equations for slab propagation.

In all other cases we must accept an approximate solution, which we take to be the one that minimizes the norm of $\bar{\bar{B}}$. Thus $k_x$ and $k_y$ are varied until $|\bar{\bar{B}}\cdot\bar{A}|^2/|\bar{A}|^2$ is minimal. It is easily shown that

$$\frac{|\bar{\bar{B}}\cdot\bar{A}|^2}{|A|^2} = \frac{\bar{A}^+\cdot\bar{\bar{B}}^+\cdot\bar{\bar{B}}\cdot\bar{A}}{\bar{A}^+\cdot\bar{A}} \geq \lambda_{\min}(k_x,k_y) \tag{39}$$

where $^+$ denotes the Hermitian conjugate and $\lambda_{\min}$ is the minimal real nonnegative eigenvalue of the Hermitian matrix $\bar{\bar{M}} = \bar{\bar{B}}^+\cdot\bar{\bar{B}}$. The equality holds when $\bar{A}$ is the appropriate eigenvector. Since $\bar{\bar{M}}$ is a $2 \times 2$ matrix, $\lambda_{\min}$ may be found directly as

$$\lambda_{\min} = \tfrac{1}{2}(m_{11} + m_{22} - ((m_{11} - m_{22})^2 + 4|m_{12}|^2)^{1/2}) \tag{40}$$

where

$$\bar{\bar{M}} = \bar{\bar{B}}^+\cdot\bar{\bar{B}} = \begin{Bmatrix} m_{11} & m_{12} \\ m_{12}^* & m_{22} \end{Bmatrix}. \tag{41}$$

The solution is then simply found by minimizing $\lambda_{\min}$ by variation of the real and imaginary parts of $k_x$ and $k_y$. In this way we have found the set of four plane waves which satisfy the boundary conditions with least squared error. Furthermore, the magnitude of $\lambda_{\min}$ is a measure of the error, or rather

$$d = \left(\frac{\lambda_{\min}}{8}\right)^{1/2} \tag{42}$$

may be taken as a measure of the rms error in (15)–(23). The minimum of $\lambda_{\min}$ as a function of $k_x,k_y$ has been found by using the minimization method of Powell [10], but in fact any nonlinear optimization method could be used. Once $k_x,k_y$ are determined, $k_z$ is given by (1). The method is fast since no matrix inversion is involved and only the minimization of an analytically given function is required.

We intend to prove that the constructed approximate solution of (16)–(23) satisfying (19), (20), and (21) exactly, is actually the optimal solution. Let $\bar{A}_i$ be any given set of fields, and let $\Delta_i$ denote the difference between the left- and right-hand sides of (16)–(23). It can be proved that

$$\sum_{i=1}^{8} |\bar{\Delta}_i|^2/|\bar{A}_3|^2 \geq \lambda_{\min}(k_x,k_y) \tag{43}$$

where $\lambda_{\min}$ is the function given in (39), and where the equality holds only for the $A$'s constructed by the method above. This means that the constructed solution minimizes the violation of the boundary conditions.

## IV. SOME EXAMPLES

### A. Dielectric Surface Waveguides

The most accurate calculations have been made by Schlosser and Unger [6], who match the fields numerically along boundaries between rectangular sections around the guide. The results are very accurate, but the method is time consuming. Schlosser and Unger therefore proposed a simpler theory, which essentially consists of solving the two-dimensional equations (37) and (38). A similar, simple method has been devised by Marcatili [1], who has also indicated an approximate analytical solution of the two-dimensional equations. The present method is more accurate than the two-dimensional methods, and simpler and faster than the point-matching schemes. A comparison with the exact results of Schlosser and Unger is made in Table I. The example concerns the fundamental $EH_{11}^y$ mode in a dielectric with $\epsilon_r = 2.5$ and $b/a = 2.5$, $\Delta$ is the relative error in percent, and $d$ is the mean error according to (42). It is rather surprising that a $d$ of 0.5 corresponds to an error of only 2 percent in $\beta_z/k_0$. Notice on the other hand that the increase of the error with decreasing frequency is natural since the simple set of plane waves cannot explain the complicated field behavior between interacting dielectric corners. The approximate theory of Schlosser and Unger (not shown in Table I) yields an error of 5 percent for $k_0a = 1.50$, and the analytical theory of Marcatili has an error of 11 percent for the same case. It should be noted that Marcatili's analytical, closed-form expression is very easy to use, and is excellent for rough estimates.

TABLE I

| $k_o a$ | Schlosser and Unger $\beta_z/k_o$ | The present theory $\beta_z/k_o$ | Δ% | d |
|---|---|---|---|---|
| 1.50 | 1.1196426 | 1.097409 | -2.0 | 0.47 |
| 2.00 | 1.2407761 | 1.238036 | -0.2 | 0.33 |
| 3.00 | 1.3833421 | 1.383341 | ~ 0 | 0.17 |
| 4.00 | 1.452385 | 1.452428 | +0.003 | 0.10 |
| 5.00 | 1.4905564 | 1.490574 | +0.001 | 0.07 |

### B. Metallic Waveguides

There is no exact theory for waveguide attenuation due to imperfectly conducting walls. Usually, by the classical method [17] the power loss is found by assuming that the surface current distribution is not disturbed by the nonzero value of the surface impedance. This is rather inaccurate for degenerate modes ($TE_{mn}$ and $TM_{mn}$; $m,n \neq 0$) in a rectangular waveguide. Van Bladel [11] and Robson [12] have found approximate expressions for the propagation characteristics of the new modes. They both assume that the absolute value of the surface impedance is small. Their results are not accurate close to cutoff, as their theories predict an infinite value of attenuation at cutoff. In the method presented here there is no assumption of a small surface impedance, and the attenuation at cutoff is finite, but in general it is difficult to say which method is the most accurate. A comparison between the various theories is given in Fig. 3, which concerns a copper guide with $a = 2.5$ cm, $b = 1.25$ cm, and the $TM_{21}$ and $TE_{21}$ modes. The names TM and TE are strictly valid in the perfectly conducting case only, since both modes are of a hybrid character. It is noted that for $f/f_c > 2$ the present method and those by van Bladel and Robson give similar results, while there is some discrepancy close to cutoff.

Another case is shown in Fig. 4 for the $TM_{11}$ and the $TE_{11}$ modes. In the case of $m$ or $n$ equal to zero the modes are not degenerate and this theory gives results which are identical to the results achieved by the classical method.

### C. Tunnel Propagation

In tunnel propagation we consider the case where the internal medium is air and the external medium a lossy dielectric. There has recently been considerable interest in mine tunnel propagation, where the experimental results of Goddard [13] have indicated that it is necessary to go into the microwave region to obtain sufficiently small attenuations. The experimental results of Goddard have been well explained by Emslie *et al.* [7], using the two-dimensional dispersion equations [(37) and (38)] asymptotically. It can easily be shown that the attenuation falls off like the frequency squared, in good agreement with experimental results. Mahmoud and Wait [8] have used a ray formulation where a source is introduced, and where the total fields are found by summing over a large number of images. In this way, the attenuation of the least attenuated mode can be found by looking at the field at a great distance from the source.

Fig. 5 shows some results typical of the lowest order modes in a tunnel. The asymptotic decay rate mentioned above is valid from about 300 MHz for the 11 modes.

It is natural that the horizontally polarized wave is the least attenuated asymptotically, since $R^+$ is effective at the smallest wall and $R^-$ at the largest; $|R^-|$ is always closer to unity than $|R^+|$. There is no sharp cutoff but instead a crossover around 100 MHz so that at lower frequencies the vertically polarized wave is the least damped.

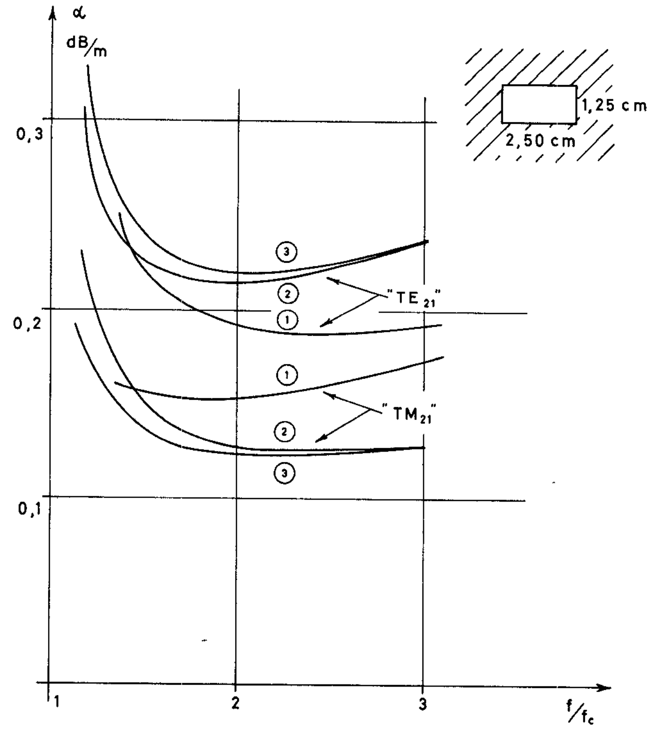


Fig. 3. Attenuation of degenerate modes in copper waveguide. $TE_{21}$,$TM_{21}$: ① classical method; ② present method; and ③ references [11],[12].

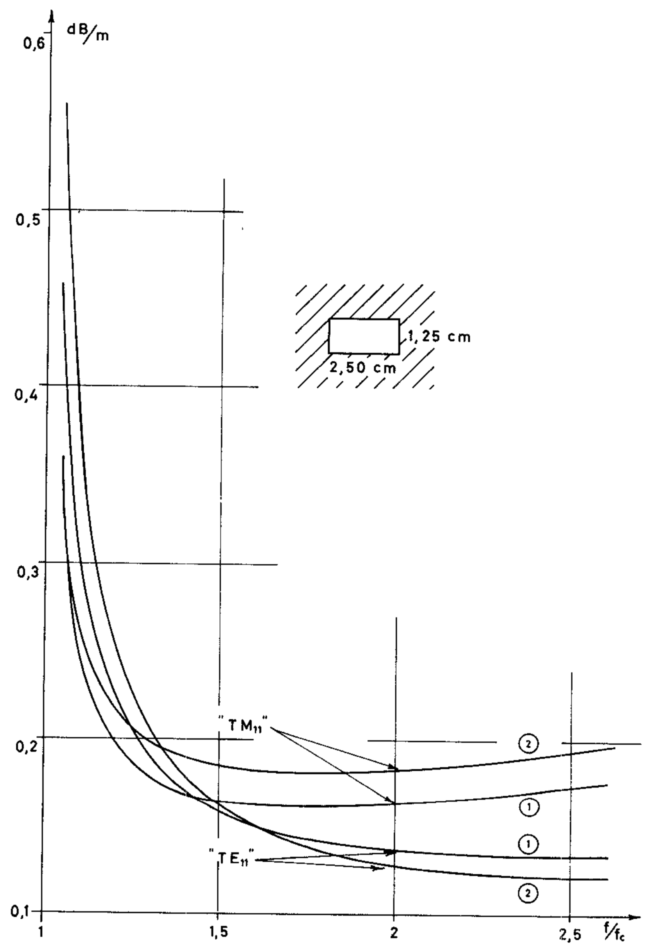


Fig. 4. Attenuation of degenerate modes in copper waveguide. $TE_{11}$,$TM_{11}$: ① classical method; and ② present method.

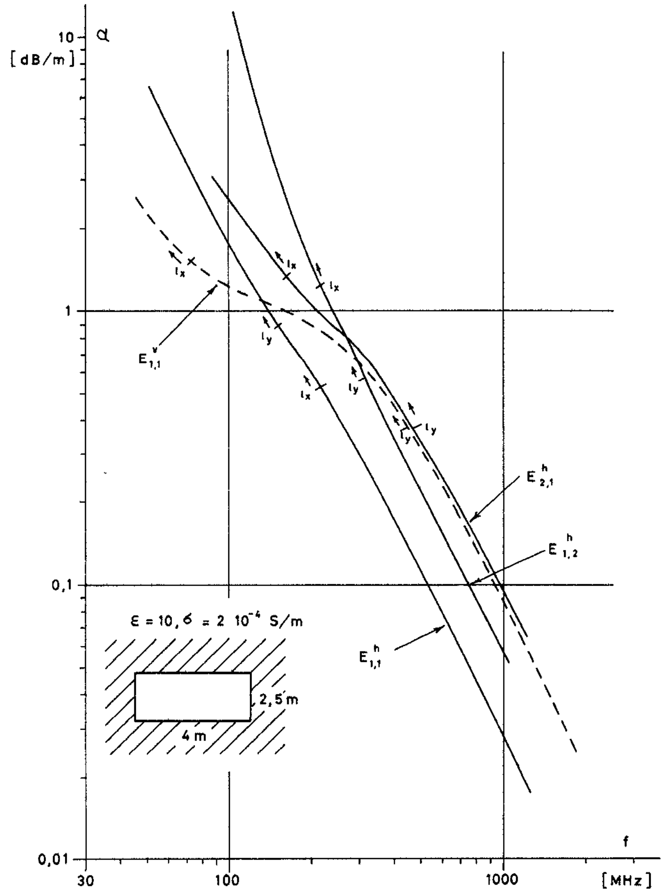


Fig. 5. Attenuation of lowest order modes in coal mine tunnel. $l_x$: leaky in horizontal direction. $l_y$: leaky in vertical direction.

The points marked $l_x$ and $l_y$ indicate another interesting transition. $l_x$ means that the wave in the external medium is leaky (grows exponentially horizontally away from the tunnel), $l_y$ means leaky in the vertical direction. If the external medium had been lossless, the waves would have been leaky for all frequencies due to refraction. When there is some finite conductivity, however, the waves will always decay exponentially transversely when the frequency is sufficiently high. On the other hand, it is noted from Fig. 5 that at the lower frequencies we have leaky waves, indicating that over some region in the medium the waves are growing exponentially, overcoming the losses. Essentially, this means that the tunnel acts as a very efficient radiator, a property which could be useful for communication through the rock.

This transverse propagation property depends heavily on the conductivity, while the attenuation along the tunnel is almost independent of the wall conductivity as long as it is small. This latter point has been noted by Glaser [14] for circular tunnels.

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